# ON A CLASS OF NONFLEXIBLE ALGEBRAS[1]

BY

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1. **Introduction.** An algebra $A$ over a field $F$ of characteristic not two will belong to the class $\mathfrak{A}$ if

I. The elements of $A$ satisfy a nontrivial identity of the form

$$\alpha_1(zx)y + \alpha_2(zy)x + \alpha_3y(zx) + \alpha_4x(zy) + \alpha_5(xz)y + \alpha_6(yz)x + \alpha_7y(xz) + \alpha_8x(yz) = 0 \tag{1}$$

for fixed $\alpha_i \in F$.

II. There is an algebra $B$ over $F$ such that $B$ satisfies (1), $B$ has an identity element, and $B$ is nonflexible; that is, there are elements $x$ and $y$ in $B$ such that $x(yx) \neq (xy)x$.

These conditions are similar to those used by Albert to define almost left alternative algebras [5]. Albert's paper led to the study of algebras of $(\gamma, \delta)$ type by Kleinfeld and Kokoris [9; 11; 12]. Kokoris has shown that any simple finite dimensional algebra of characteristic prime to 30 of $(\gamma, \delta)$ type is either alternative or has an identity element which is an absolutely primitive idempotent[2]. Our alteration of Albert's conditions yields a new class of simple power-associative algebras. We note that property II seems more natural in light of Oehmke's results [13] and the remark that most of the well-known nonassociative algebras (Jordan, noncommutative Jordan, Lie, alternative, associative) satisfy the flexible identity $x(yx) = (xy)x$.

In §2 it is shown that if $F$ is algebraically closed then any algebra $A$ over $F$ belonging to $\mathfrak{A}$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ where $A(\mu)$ satisfies one of the following identities:

(i) $(xy)z - x(yz) = (zy)x - z(yx)$,

(ii) $x(xy) + (yx)x = 2(xy)x$,

(iii) $x(xy) + (yx)x = (xy)x + x(yx)$,

(iv) $x(yz + zy) + (yz + zy)x = (xy + yx)z + z(xy + yx)$.

The remainder of the paper is devoted to the study of algebras which satisfy some one of these four identities. We find that any power-associative

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(2) For results on rings of $(\gamma, \delta)$ type see a forthcoming paper of Kleinfeld to appear in the Canadian Journal of Mathematics.

ring $A$ which satisfies (i) has an idempotent decomposition as $A=A_{11}+A_{10}+A_{01}+A_{00}$ where the $A_{ij}$ are defined just as in the associative case. Using this decomposition, we prove in §3 that any simple power-associative ring of characteristic not two which satisfies (i) and has an idempotent $e$ such that $A_{10}+A_{01}\neq 0$ is an associative ring. Examples of simple power-associative algebras satisfying (i) which are not flexible are constructed.

In §4 we are able to use Oehmke's methods [13] to prove that any semi-simple strictly power-associative algebra $A$ over a field $F$ of characteristic prime to 6 which satisfies (ii) has an identity and is the direct sum of simple algebras. The main result on algebras satisfying (ii) is that any simple strictly power-associative algebra of characteristic prime to 6 of degree $t>2$ is flexible so that the results of [13] yield the result that $A$ is one of the following:

(a) a commutative Jordan algebra;

(b) a quasi-associative algebra;

(c) an algebra of degree 1 or 2.

Finally, examples of simple power-associative nonflexible algebras which satisfy either (iii) or (iv) are constructed.

As a matter of terminology, by an algebra we shall always mean a finite dimensional vector space on which there is a multiplication defined which satisfies both distributive laws. The radical of a power-associative ring is the maximal nil ideal and any ring with zero radical is said to be semi-simple. If $A$ is any power-associative ring of characteristic not two in which the equation $2x=a$ has a solution for all $a\in A$, then $A$ has an attached ring $A^{(+)}$ which is the same additive group as $A$ but the multiplication $x\circ y$ of $A^{(+)}$ is defined by $2x\circ y=xy+yx$. Then $A$ has a decomposition with respect to an idempotent $e$ as

$$A = Ae(2) + Ae(1) + Ae(0) \text{ where } x\in Ae(\lambda) \text{ if and only if } 2e\circ x = \lambda x, \quad \lambda = 0, 1, 2 \text{ [3]}.$$

2. **The Class $\mathfrak{A}$.** By Property II there is an algebra $B$ in $\mathfrak{A}$ with elements $x$ and $y$ such that $xy\neq yx$. Then, by a series of substitutions of the elements 1, $x$, $y$ in (1), we find the following relations:

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8 = 0,$$
$$\alpha_1+\alpha_4+\alpha_5+\alpha_8 = \alpha_2+\alpha_3+\alpha_6+\alpha_7 = 0,$$
$$\alpha_1+\alpha_2+\alpha_3+\alpha_6 = \alpha_4+\alpha_5+\alpha_7+\alpha_8 = 0,$$
$$\alpha_1+\alpha_2+\alpha_4+\alpha_5 = \alpha_3+\alpha_6+\alpha_7+\alpha_8 = 0.$$

Combining these we have $\alpha_8=\alpha_2$, $\alpha_7=\alpha_1$, $\alpha_6=-(\alpha_1+\alpha_2+\alpha_3)$, and $\alpha_5=-(\alpha_1+\alpha_2+\alpha_4)$.

For an arbitrary ring $A$ we define $R_x$ to be the mapping $a\rightarrow ax$ and $R_x$ is called a right multiplication of $A$. Similarly $L_x$ is defined as the mapping $a\rightarrow xa$ and is called a left multiplication of $A$.

Now using these relations and rewriting (1) in terms of right and left multiplications, we obtain

$$(2)\qquad \begin{aligned}\alpha_1(R_xR_y + L_xL_y) + \alpha_2(R_yR_x + L_yL_x) + \alpha_3R_xL_y + \alpha_4R_yL_x\\ - (\alpha_1 + \alpha_2 + \alpha_4)L_xR_y - (\alpha_1 + \alpha_2 + \alpha_3)L_yR_x = 0.\end{aligned}$$

If we interchange $x$ and $z$ in (1), we find

$$(3)\qquad \begin{aligned}\alpha_1(L_xR_y + R_xL_y) + \alpha_2(L_{xy} + R_{yx}) + \alpha_3L_xL_y + \alpha_4R_{xy}\\ - (\alpha_1 + \alpha_2 + \alpha_4)R_xR_y - (\alpha_1 + \alpha_2 + \alpha_3)L_{yx} = 0.\end{aligned}$$

Finally, setting $y=x$ in (2) yields

$$(4)\qquad (\alpha_1 + \alpha_2)(R_x^2 + L_x^2 - 2L_xR_x) + (\alpha_3 + \alpha_4)(R_xL_x - L_xR_x) = 0.$$

Suppose $\alpha_1+\alpha_2=0$. Then from (4) we must have $(\alpha_3+\alpha_4)(R_xL_x-L_xR_x)=0$, but property II implies there is an $x\in B$ such that $R_xL_x-L_xR_x\neq 0$, so that $\alpha_3+\alpha_4=0$. Suppose also that $\alpha_1=\alpha_2=0$. Substitution of these values in (3) along with the condition that not all the $\alpha_i$ are zero yields

$$(5)\qquad R_{xy} - R_xR_y = -(L_{yx} - L_xL_y).$$

Suppose $A$ and $B$ are algebras over a field $F$ such that $A$ and $B$ are isomorphic as vector spaces. We may then consider $A$ and $B$ as the same vector space and we shall say that $A$ is quasi-equivalent in $F$ to $B=A(\mu)$ if there is a $\mu\in F$, $\mu\neq 1/2$, such that the product $x\circ y$ in $B$ is given in terms of the product $xy$ of $A$ by $x\circ y=\mu xy+(1-\mu)yx$ [3]. The multiplications $R_x'$ and $L_x'$ of $A(\mu)$ are given in terms of the multiplications $R_x$ and $L_x$ of $A$ by $R_x'=\mu R_x+(1-\mu)L_x$ and $L_x'=(1-\mu)R_x+\mu L_x$.

In the remainder of this section we suppose $A$ to belong to the class $\mathfrak{A}$ and $F$ to be an algebraically closed field.

LEMMA 2.1. *Suppose $\alpha_1+\alpha_2=0$ and $\alpha_1\neq\pm\alpha_3$. Then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ satisfying identity* (5).

**Proof.** If $\alpha_1=0$ the result is immediate. If $\alpha_1\neq 0$ and $\beta=-\alpha_3/\alpha_1\neq\pm 1$, then (3) can be written as

$$(6)\qquad L_{xy} - L_xR_y - R_xL_y + R_{yx} = \beta(R_{xy} - R_xR_y - L_xL_y + L_{yx}).$$

We note that $R'_{x\circ y}-R_x'R_y'-L_x'L_y'+L'_{y\circ x}=0$ in $A(\mu)$ is equivalent to $\mu^2R_{xy}+\mu(1-\mu)L_{xy}+\mu(1-\mu)R_{yx}+(1-\mu)^2L_{yx}-(\mu R_x+(1-\mu)L_x)(\mu R_y+(1-\mu)L_y)-((1-\mu)R_x+\mu L_x)((1-\mu)R_y+\mu L_y)+\mu(1-\mu)R_{yx}+\mu^2L_{yx}+(1-\mu)^2R_{xy}+\mu(1-\mu)L_{xy}=0$ in $A$. Simplifying we have

$$(7)\qquad \begin{aligned}(2\mu^2 - 2\mu + 1)(R_{xy} - R_xR_y - L_xL_y + L_{yx})\\ = -2\mu(1-\mu)(L_{xy} - L_xL_y - R_xL_y + R_{yx}).\end{aligned}$$

Now consider the equation in $\mu$, $2\mu^2-2\mu+1=-2\mu(1-\mu)\beta$ or $2(1-\beta)\mu^2-2(1-\beta)\mu+1=0$. Since $\beta\neq 1$ and $F$ is algebraically closed, there is a $\mu$ in $F$ satisfying this equation. Suppose $\mu=1/2$. Then the above equation becomes $1-\beta-2(1-\beta)+2=0$. Hence, $\beta=-1$ contrary to assumption. Then substitution of this value of $\mu$ in (7) yields (6) and the proof is complete.

Let $\alpha_1=-\alpha_3$ and $\alpha_1+\alpha_2=0$. Then (6) becomes $L_{xy}-L_xR_y-R_xL_y+R_{yx}=R_{xy}-R_xR_y-L_xL_y+L_{yx}$ or

$$L_{xy-yx} - R_{xy-yx} = L_xR_y - R_xR_y - L_xL_y + R_xL_y. \tag{8}$$

If $\alpha_1=\alpha_3$ and $\alpha_1+\alpha_2=0$, we see that (6) becomes

$$\begin{aligned} R_{xy+yx} + L_{xy+yx} &= (R_x + L_x)(R_y + L_y) \\ \text{(by symmetry)} &= (R_y + L_y)(R_x + L_x). \end{aligned} \tag{9}$$

This is exactly the condition that $A$ be associative-admissible.

LEMMA 2.2. *If $\alpha_1+\alpha_2\neq 0$, then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ satisfying either*

$$R_x^2 + L_x^2 = 2L_xR_x \quad or \tag{10}$$

$$R_x^2 + L_x^2 = L_xR_x + R_xL_x. \tag{11}$$

**Proof.** Since $\alpha_1+\alpha_2\neq 0$, we may write (4) in the form

$$R_x^2 + L_x^2 - 2L_xR_x = \beta(L_xR_x - R_xL_x) \quad \text{where} \quad \beta = \frac{\alpha_3 + \alpha_4}{\alpha_1 + \alpha_2}. \tag{12}$$

Then as before, $(R_x')^2 + (L_x')^2 - 2L_x'R_x' = 0$ in $A(\mu)$ is equivalent to $(\mu R_x+(1-\mu)L_x)^2+(\mu L_x+(1-\mu)R_x)^2-2(\mu L_x+(1-\mu)R_x)(\mu R_x+(1-\mu)L_x)=(4\mu^2-4\mu+1)(R_x^2+L_x^2-2L_xR_x)-(4\mu^2-6\mu+2)(R_xL_x-L_xR_x)=0$ in $A$. Now examine the equation $(4\mu^2-4\mu+1)\beta=-(4\mu^2-6\mu+2)$ or $(2\mu-1)((2+2\beta)\mu-(2+\beta))=0$. This has a solution $\mu\neq 1/2$ provided $\beta\neq -1$. Thus the theorem is valid except possibly when $\beta=-1$. But then (12) becomes $R_x^2+L_x^2=L_xR_x+R_xL_x$.

Linearization of (10) gives us

$$R_xR_y + R_yR_x + L_xL_y + L_yL_x = 2L_xR_y + 2L_yR_x \tag{13}$$

which is equivalent to (10) since the characteristic of $F$ is not two. Similarly (11) is equivalent to

$$R_xR_y + R_yR_x + L_xL_y + L_yL_x = R_xL_y + R_yL_x + L_xR_y + L_yR_x \tag{14}$$

provided the characteristic of $F$ is not two. We note that setting $y=x$ in (8) yields (11). Combining these remarks with Lemmas 2.1 and 2.2 we state

THEOREM 2.1. *Let $A$ belong to the class $\mathfrak{A}$ and suppose also that $F$ is algebraically closed. Then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ where $A(\mu)$*

*satisfies one of the identities* (5), (9), (10), (11); *each of which is a particular determination of* (1).

3. **The identity** $R_{xy}-R_xR_y=-(L_{yx}-L_xL_y)$. In the following we shall be concerned with a ring $A$ of characteristic not two such that $A$ satisfies identity (5). Set $(x, y, z)=(xy)z-x(yz)$. $(x, y, z)$ is called an associator. Then (5), when applied to an element $z$ of $A$, becomes in terms of associators

$$(x, y, z) = (z, y, x). \tag{15}$$

We now write the linearization of $xx^2=x^2x$ [3] in terms of associators as

$$(x, y, z) + (x, z, y) + (y, x, z) + (y, z, x) + (z, x, y) + (z, y, x) = 0. \tag{16}$$

Using (15) in (16) we obtain

$$(x, y, z) + (y, z, x) + (z, x, y) = 0. \tag{17}$$

THEOREM 3.1. *If $A$ is a ring of characteristic zero satisfying* (15) *and $xx^2=x^2x$, then $A$ is power-associative.*

**Proof.** By [1, Lemma 4], we need only show that $x^2x^2=x^3x$ for all $x\in A$. Set $y=x$ and $z=x^2$ in (15) and (17). Then $(x, x, x^2)=(x^2, x, x)$ and $(x, x, x^2)+(x, x^2, x)+(x, x, x^2)=0$. Thus $2x^2x^2=x^3x+xx^3$ and $2x^3x-2xx^3=0$, so that $x^2x^2=x^3x=xx^3$ and the result follows.

The following example shows that $xx^2=x^2x$ is actually necessary to guarantee power-associativity. Let $A$ be the algebra with a basis of the three elements $e$, $u$, and $v$ over a field $F$ of characteristic zero where multiplication is defined by $uv=e$, $eu=u$, $ve=v$, $e^2=e$, and all other products zero. Then $A$ satisfies (5), but we observe that $(u+v)^2(u+v)=u$ and $(u+v)(u+v)^2=v$.

We now write the flexible identity $x(yx)=(xy)x$ in its linearized form as

$$x(yz) + z(yx) = (xy)z + (zy)x. \tag{18}$$

The flexible identity and (18) are equivalent provided $A$ has characteristic different from 2.

We note that (15) along with (18) implies associativity while (15) and (17) together yield $2(x, x, y)=2(y, x, x)=-(x, y, x)$. Hence, for rings of characteristic not two satisfying (15) and $xx^2=x^2x$, the notions of being associative, flexible, right alternative, and left alternative are equivalent.

Hereafter, whenever we refer to a substitution or a permutation of certain elements in an identity given in terms of the right and left multiplications of the elements $x$, $y$, $xy$, and $yx$, we shall consider these multiplications as acting on the element $z$. Also, we shall often make use of these identities without writing them in terms of right and left multiplications.

A ring $A$ is said to be Jordan-admissible if the attached ring $A^{(+)}$ is a Jordan ring. A ring $A$ is called Lie-admissible if the ring $A^{(-)}$, where $A^{(-)}$ is the same additive group as $A$ but $A^{(-)}$ has the multiplication $[x, y]=xy-yx$, is a Lie ring.

THEOREM 3.2. *A is both Jordan-admissible and Lie-admissible.*

**Proof.** Permuting $x$, $y$, $z$ in (5) we obtain

$$R_{yx} - R_yR_x - L_yL_x + L_{xy} = 0 \tag{19}$$

and

$$R_xL_y - R_yL_x - L_yR_x + L_xR_y = 0. \tag{20}$$

Subtracting (5) from (19) and then adding (20) to the resulting right hand member we have $L_{xy-yx} - R_{xy-yx} = (R_y - L_y)(R_x - L_x) - (R_x - L_x)(R_y - L_y)$, so that $A$ is Lie-admissible [3].

Now set $y = x^2$ in (5). Then

$$R_{x(xx)} + L_{(xx)x} = R_xR_{xx} + L_xL_{xx} \tag{21}$$

and $y = x^2$ in (19) yields

$$R_{(xx)x} + L_{x(xx)} = R_{xx}R_x + L_{xx}L_x. \tag{22}$$

We also set $y = x^2$ in (20) to obtain

$$R_xL_{xx} - R_{xx}L_x - L_{xx}R_x + L_xR_{xx} = 0. \tag{23}$$

Subtracting (22) from (21) and then adding (23) we obtain

$$(R_x + L_x)(R_{xx} + L_{xx}) - (R_{xx} + L_{xx})(R_x + L_x) = 0$$

and the theorem is proved [3].

In the remainder of this section we suppose that $A$ is a power-associative ring.

LEMMA 3.1. *If $e$ is any idempotent of $A$, then $(e, e, x) = (x, e, e) = (e, x, e) = 0$ for all $x \in A$.*

**Proof.** If we set $A = A_e(2) + A_e(1) + A_e(0)$, we then see that it is sufficient to prove the proposition for $x \in A_e(1)$.(3) Thus, we suppose in the following that $x \in A_e(1)$. Substitution of $y = z = e$ in (15) yields

$$ex + xe = e(ex) + (xe)e, \tag{24}$$

and since $x = ex + xe = e(ex) + (xe)e + e(xe) + (ex)e$ we have

$$(ex)e + e(xe) = 0. \tag{25}$$

Set $ex = x_2 + x_1 + x_0$ and $xe = -x_2 + x_1' - x_0$ where $x_i \in A_e(i)$. Then by substitution in

$$ex = e(ex + xe) = e(ex) + e(xe) = e(ex) - (ex)e \tag{26}$$

we find $x_1e = -(x_2 + x_0)/2$, so that $ex_1 = (x_2 + x_0)/2 + x_1$. But then $x_1 = ex_1 + x_1e$

(3) This decomposition of power-associative rings is basic to our development so that we refer the reader to [2, Chapter I, Theorem 3].

$=e(ex_1)+(x_1e)e=x_2/2+ex_1-x_2/2=ex_1$. Hence, $ex_1=x_1$ and $x_2=x_0=0$, so that by consideration of (24), (25), and (26), we have $e(ex)-ex=xe-(xe)e=e(xe)-(ex)e=0$ and the result follows.

THEOREM 3.3. *If $e$ is any idempotent of $A$, then every $x\in A$ may be written uniquely as $x=x_{11}+x_{10}+x_{01}+x_{00}$ where $x_{ij}\in A_{ij}=\{a\colon ea=ia;\ ae=ja\}$, $i, j=0, 1$.*

**Proof.** The theorem follows immediately from Lemma 3.1 just as in the associative case.

Let $x$ and $y\in A_{11}$. Then from (15) $e(xy)+(yx)e=(ex)y+y(xe)=xy+yx$. If $xy=a_{11}+a_{10}+a_{01}+a_{00}$, then $yx=b_{11}-a_{10}-a_{01}-a_{00}$ so that $a_{10}-a_{01}=0$, and hence, $a_{10}=a_{01}=0$. Therefore $A_{11}^2\subseteq A_{11}+A_{00}$. If $x, y\in A_{00}$, then $e(xy)+(yx)e=(ex)y+y(xe)=0$, and, as above, we find $A_{00}^2\subseteq A_{11}+A_{00}$. If $x\in A_{11}$, $y\in A_{10}$, then $xy+yx=(xe)y+y(ex)=x(ey)+(ye)x=xy$. Thus $yx=0$. But then $e(xy)+(yx)e=(ex)y+y(xe)$ or $e(xy)=xy$ and $e(yx)+(xy)e=(ey)x+x(ye)=0$, so that $(xy)e=0$. Hence $A_{11}A_{10}\subseteq A_{10}$ and $A_{10}A_{11}=0$. Replacing $y\in A_{10}$ by $y\in A_{01}$, we find that $A_{11}A_{01}=0$ and $A_{01}A_{11}\subseteq A_{01}$. Then replacing $x\in A_{11}$ by $x\in A_{00}$, we have $A_{00}A_{10}=A_{01}A_{00}=0$, $A_{10}A_{00}\subseteq A_{10}$, and $A_{00}A_{01}\subseteq A_{01}$. Suppose $x, y\in A_{10}$. Then $yx=(xe)y+y(ex)=x(ey)+(ye)x=xy$ and $e(xy)+(yx)e=(ex)y+y(xe)=xy$. Hence $e(xy)+(xy)e=xy$, so that $xy\in A_{10}+A_{01}$. But $2xy=xy+yx\in A_{11}+A_{00}$. Thus $xy=yx=0$. Likewise $A_{01}^2=0$. Now suppose $x\in A_{10}$, $y\in A_{01}$. Then $e(xy)+(yx)e=(ex)y+y(xe)=xy$ and $e(yx)+(xy)e=(ey)x+x(ye)=xy$. If we set $xy=a_{11}+a_{10}+a_{01}+a_{00}$ and $yx=b_{11}-a_{10}-a_{01}+b_{00}$ and substitute in the above equations, we find $a_{11}+a_{10}+b_{11}-a_{01}=a_{11}+a_{10}+a_{01}+a_{00}=b_{11}-a_{10}+a_{11}-a_{01}$. Hence $b_{11}=a_{01}=a_{00}=a_{10}=0$ so that $xy=a_{11}$ and $yx=b_{00}$. Combining these results we state

THEOREM 3.4. *Suppose $e$ is any idempotent of the ring $A$ and $A=A_{11}+A_{10}+A_{01}+A_{00}$ where the $A_{ij}$ are defined as in Theorem 3.3. If $i$ and $j$ are distinct and $i, j, k, m=0, 1$, then $A_{ii}^2\subseteq A_{ii}+A_{jj}$, $A_{kj}A_{im}=0$, and if either $k\neq j$, or $j\neq m$ then also $A_{kj}A_{jm}\subseteq A_{km}$.*

COROLLARY 3.1. *$L=A_{10}A_{01}+A_{10}+A_{01}+A_{01}A_{10}$ is an ideal of $A$.*

**Proof.** By the preceding theorem it is sufficient to show that $A_{ii}(A_{ij}A_{ji})+(A_{ij}A_{ji})A_{ii}\subseteq L$ for $i\neq j$. But if $x\in A_{ii}$, $y\in A_{ij}$, $z\in A_{ji}$, then $x(yz)=(xy)z+z(yx)-(zy)x=(xy)z$. Then $x(yz)\in A_{ij}A_{ji}\subseteq L$. Likewise $(A_{ij}A_{ji})A_{ii}\subseteq L$.

COROLLARY 3.2. *Let $A$ be simple and $e$ an idempotent of $A$. Then either $A_{10}+A_{01}=0$ or $A_{11}=A_{10}A_{01}$ and $A_{00}=A_{01}A_{10}$.*

**Proof.** The result is immediate from Corollary 3.1.

We now state the main result of this section.

THEOREM 3.5. *If $A$ is a simple power-associative ring of characteristic not two possessing an idempotent $e$ such that $A_{10}+A_{01}\neq 0$ and satisfying identity (21), then $A$ is associative.*

**Proof.** Let $x \in A_{ij}$, $y \in A_{rs}$, $z \in A_{mk}$. Then $x(yz)=0$ unless $s=m$ and $j=r$. Similarly $(xy)z=0$ unless $s=m$ and $j=r$. Hence, it is sufficient to consider products of the form $x(yz)$ where $x \in A_{ij}$, $y \in A_{js}$, and $z \in A_{sk}$.

*Case* 1. Suppose $i=j=s=m$. We may represent $x$ as $x=\sum x_{in}x_{ni}$ where $i \neq n$ since we may use Corollary 3.2. Then, by repeated applications of identity (15) and Theorem 3.4 we find

$$(xy)z = \sum ((x_{in}x_{ni})y)z = \sum (x_{in}(x_{ni}y))z = \sum x_{in}((x_{ni}y)z)$$
$$= \sum x_{in}(x_{ni}(yz)) = \sum (x_{in}x_{ni})(yz) = x(yz).$$

*Case* 2. Let $i=j$. Then $x(yz)=(xy)z+z(yx)-(zy)x$ and the product is associative unless $i=m=s$ since otherwise the last two terms of the right hand member are zero. But, if $i=m=s$, Case 1 applies so that the result holds in this instance.

*Case* 3. Suppose $j=s$. Proceed as in Case 2.

*Case* 4. Suppose $s=m$. Proceed as in Case 2.

Thus we have reduced the proof to the case where $x$, $z \in A_{ij}$, $y \in A_{ji}$, with $i \neq j$. Substitution of $x$, $y$, and $z$ in (16) along with Theorem 3.4 yields

$$x(yz) + z(yx) = (xy)z + (zy)x \tag{27}$$

which with (15) implies $x(yz)=(xy)z$ and the proof is complete.

REMARK. If $A$ is a semi-simple strictly power-associative algebra of characteristic not two satisfying (15) and $e$ is a principal idempotent of $A$, then $A_{10}+A_{01}=0$.

**Proof.** We observe that $e$ is also a principal idempotent of $A^{(+)}$ and $A_e(1)=A_{10}+A_{01}$. Hence $A_{10}+A_{01}+A_{00} \subseteq$ radical of $A^{(+)}$ [10, Theorem 5]. Then we claim that the ideal $L$ defined in Corollary 3.1 of Theorem 3.4 is contained in the radical of $A^{(+)}$. For, if $x \in A_{10}$, $y \in A_{01}$, then $2x \circ y = xy+yx \in$ radical of $A^{(+)}$. But $2e(x \circ y)+2(x \circ y)e = 2xy \in$ radical of $A^{(+)}$. Thus $A_{10}A_{01} \subseteq$ radical of $A^{(+)}$, so that $L$ must be a nil ideal of $A$. Therefore $A_{10}+A_{01}=0$.

Theorem 3.5 implies that if we are to find any new simple power-associative algebras satisfying (15), they must have no idempotent $e$ such that $A_{10}+A_{01} \neq 0$. The existence of such algebras is guaranteed by the following examples. First, let $A$ be the 3-dimensional algebra over a field of characteristic $p \neq 2$ with a basis $e$, $u$, $v$ where $e$ is the identity element, $uv=-vu=e$, and the remaining products are zero. Then $A$ is a simple power-associative algebra satisfying (15) but $A$ is not flexible for $(uv)u=-u(vu)=u$. Suppose we set $B$ equal to the supplementary sum of the two orthogonal subspaces $A_i$, $i=1, 2$; where $A_i$ has a basis $e_i$, $u_i$, $v_i$ such that $e_i$ is the unity of $A_i$, $u_i^2=v_i^2=0$, and $u_iv_i=-v_iu_i=e_1+e_2=e$ the unity element of $B$. Then $B$ is a simple power-associative algebra of degree two satisfying (15) and, as before, $B$ is not flexible. This construction is easily generalized to yield simple power-associative algebras of arbitrary degree $n$ which satisfy (15) but not the flexible identity.

We now give an example of a semi-simple power-associative algebra $C$ which satisfies (15) but has no identity element and is not the direct sum of simple algebras. Consider $C$ equal to the supplementary sum of the algebra $A$ above and the subspace $A'$ with a basis $u_1$, $v_1$ where $AA'=A'A=0$ and $u_1v_1=-v_1u_1=e$ and all other products zero. Then, since every proper ideal of $C$ contains $A$, we see that the nil radical of $C$ is 0.

One might hope for further results when $A$ is semi-simple with an idempotent $e$ such that $A_{10}+A_{01}\neq 0$, but examination of the algebra $D=C\oplus F_2$, where $C$ is the algebra described just above and $F_2$ is the total matric algebra over $F$ of degree 2, shows there are such algebras which do not have an identity and are not the direct sum of simple algebras.

4. **The identity $R_x^2+L_x^2=2L_xR_x$.** In this section we assume $A$ to be a ring having characteristic prime to six and having a solution to the equation $2x=a$ for all $a\in A$. Moreover, we shall assume that $A$ satisfies (13) or equivalently (10). We write (13) when applied to an element $z$ of $A$ as

$$(zx)y + (zy)x + y(xz) + x(yz) = 2(xz)y + 2(yz)x. \tag{28}$$

THEOREM 4.1. *Suppose $A$ is a ring of characteristic prime to* 30 *such that $A$ satisfies* (28) *and $(x^2)^2=(x^2x)x$ for all $x\in A$. Then $A$ is power-associative.*

**Proof.** We define $x^n$ inductively by $x^{n+1}=x^nx$. First we show that $xx^n=x^nx$ for all $x\in A$ and all positive integers $n$. Substitution of $x=y=z$ in (28) yields $x^2x+xx^2=2x^2x$. Hence $xx^2=x^2x$. Suppose $xx^k=x^kx$. Then setting $x=z$ and $y=x^k$ in (28) yields $(x^kx)x+x(xx^k)=2(xx^k)x$ so that $xx^{k+1}=x^{k+1}x$. Thus $xx^n=x^nx$ for all $x\in A$ and all positive integers $n$. The result then follows from [1, Lemma 4].

We now suppose that $A$ is a power-associative ring. We shall make frequent use of the identity $(x^2)^2=(x^2x)x$ in its linearized form

$$\sum_6 (xy + yx)(zw + zw) = \sum_4\left[\sum_3 (xy + yx)w\right]z \text{ (symmetric in } x, y, z, w) \tag{29}$$

and we note that (29) is equivalent to $(x^2)^2=(x^2x)x$ if $A$ has characteristic prime to 6. Let $e$ be an idempotent of $A$. Then from (28) we find

$$(xe)e + e(ex) = 2(ex)e \qquad \text{for all } x \in A. \tag{30}$$

Suppose $x\in A_e(1)$. Then by consideration of $A^{(+)}$ we may set $ex=x_2+x/2+x'+x_0$ and $xe=-x_2+x/2-x'-x_0$ where $x'\in A_e(1)$. Substitution in (30) yields $-x_2+xe/2-x'e-ex/2+ex'=2x_2+xe+2x'e$ so that we have $x_2-x'-x_0=ex'-3x'e$. Hence $x_2=4(ex')_2$, $x_0=4(ex')_0$, and $x'=3(x'e)_1-(ex')_1=ex'+x'e$. Thus $(ex')_1=(x'e)_1=x'$. If we now carry through the same argument with $x$ replaced by $x'$, we find that $(ex')_2=(ex')_0=0$ and hence, $x_2=x_0=0$. We note that $e(xe)=(ex)e=x/4$. Therefore, for any $x\in A_e(1)$, we have $ex=x/2+x'$ where $x'\in A_e(1)$ and $ex'=x'e=x'/2$.

Suppose $x, y\in A_e(2)$. Then we set $xy=a_2+a_1+a_0$, $yx=b_2-a_1-a_0$. Setting $y=e$, $z=y$, (28) yields $xy+yx+(yx)e+e(xy)=2yx+2(xy)e$ so that $ea_1-a_1e=-2a_1-2a_0+2a_1e$. Hence $a_0=0$ and $2a_1=3a_1e-ea_1=2a_13+2ea_1$. But then $3ea_1=a_1e=3a_1/4$. For any $a_1\in A_e(1)$ we have $ea_1=a_1/2+a'$ where $ea'=a'e=a'/2$ so that $a_1/2-a_1'=3a_1/4$. Then $-4a_1'=a_1$ and therefore $ea_1=a_1e$. Thus we finally have $4a_1'=a_1=0$ and $A_e(2)$ is a subring of $A$.

Let $x, y\in A_e(0)$. Then substitution of $y=e$, $z=y$ in (28) yields $(yx)e+e(xy)=2(xy)e$. We suppose $xy=a_2+a_1+a_0$ and $yx=-a_2-a_1+b_0$. Then $-a_1e+ea_1=2a_2+2a_1e$. Hence $a_2=0$ and $ea_1=3a_1e$. Then, just as for $A_e(2)$, we find $a_1=0$ and $A_e(0)$ is a subring of $A$.

Suppose $x\in A_e(2)$, $y\in A_e(1)$. Then let $xy=a_2+a_1+a_0$, $yx=-a_2+b_1+b_0$. Then (28) with $x=e$, $z=x$ becomes $(xy)e+xy+e(yx)+yx=2(yx)e+2xy$ and, hence, $a_1e+eb_1+b_1+b_0=2b_1e+a_1+a_0$. Thus $a_0=b_0$ and $a_1e+eb_1=2b_1e+a_1-b_1$ or $ea_1=2b_1-3b_1e$. Another substitution of $y=e$, $z=y$ in (28) yields $(yx)e+(ye)x+e(xy)+x(ey)=2(xy)e+2(ey)x$. Now set $ey=y/2+y'$ where $ey'=y'e=y'/2$. Then $b_1e+ea_1+a_1/2+xy'=a_2+2a_1e+b_1/2+2y'x$. Comparing components we find $a_2=(xy'-3y'x)_2=4(xy')_2$ and $(y'x)_0=(xy')_0=0$. Yet another substitution of $y=y'$, $z=e$ in (28) yields $xy'=y'x$ so that $(xy')_2=(y'x)_2=0$. Thus $a_2=0$. Since $ea_1=2b_1-3b_1e$ a substitution of $eb_1=b_1/2+b_1'$ and $ea_1=a_1/2+a_1'$ gives the relation $(b_1-a_1)/2=a_1'-3b_1'$. Then $e(b_1-a_1)=(b_1-a_1)e=(b_1-a_1)/2$. If $ea_1=2b_1-3b_1e$, then $e(ea_1)=2eb_1-3e(b_1e)$. But then we obtain $(b_1-a_1)/2=(2a_1'-4b_1')$ so that $a_1'=b_1'$. Finally, $eb_1-a_1e=(b_1-a_1)/2+2a_1'=0$.

Let $x\in A_e(0)$, and $y\in A_e(1)$. We set $xy=a_2+a_1+a_0$ and $yx=b_2+b_1-a_0$. Substitution of $x=e$, $z=x$ in (28) yields $a_2+a_1e+eb_1=b_2+2b_1e$. Then $a_2=b_2$ and $a_1e=2b_1e-eb_1$. Set $ey=y/2+y'$. Then, as before, $y'x=xy'$ and substitution of $z=e$ in (28) yields $(xy-yx)/2=-2xy'$ so that $a_0=0$. Now $a_1e=2b_1e-eb_1$ implies $e(a_1e)=2e(b_1e)-e(eb_1)$ or $(b_1-a_1)/4=b_1'$. We shall use these relations in our later work.

Thus we may state

THEOREM 4.2. *Let $e$ be an idempotent of a power-associative ring $A$ which satisfies* (28). *Then $A_e(2)$ and $A_e(0)$ are orthogonal subrings of $A$ and*

$$A_e(\lambda)A_e(1) + A_e(1)A_e(\lambda) \subseteq A_e(2-\lambda) + A_e(1), \qquad \lambda = 0, 2.$$
$$eA_e(1) + A_e(1)e \subseteq A_e(1).$$

Suppose $A$ has an identity element $1=e_1+\cdots+e_t$ where $e_i$ are pairwise orthogonal idempotents. Then $A$ can be decomposed as the direct sum of the modules $A_{ij}$ where for $i=j$, $A_{ij}=A_{e_i}(2)$ and for $i\neq j$, $A_{ij}=A_{e_i}(1)\cap A_{e_j}(1)$ [6]. Let $i, j, m, n$ be distinct. Then $A_{ij}\circ A_{mj}\subseteq A_{im}$, $A_{ij}\circ A_{ij}\subseteq A_{ii}+A_{jj}$, $A_{ij}\circ A_{mn}=0$. But $A_{mn}\subseteq A_{e_m}(1)$, $A_{ij}\subseteq A_{e_m}(0)$ so that by our multiplication $A_{ij}A_{mn}\subseteq A_{e_m}(1)+A_{e_m}(2)$. Since the $A_{e_m}(2)$ components of $a_{ij}a_{mn}$ and $a_{mn}a_{ij}$ are the same, $A_{ij}A_{mn}\subseteq A_{e_m}(1)$. Likewise $A_{ij}A_{mn}\subseteq A_{e_r}(1)$ for $r=i, j, n$. Thus

$A_{ij}A_{mn}=0$. Since $A_{mn}\subseteq A_{e_i}(0)$ we have $A_{ii}A_{mn}=A_{mn}A_{ii}=0$. Again, these results will be of use to us in later developments.

A ring is said to be stable if for every idempotent $e$ of $A$ we have $A_e(\mu)A_e(1)+A_e(1)A_e(\mu)\subseteq A_e(1)$ for $\mu=0$, 2. By our multiplication it is readily seen that a ring $A$ satisfying (28) is stable if and only if $A^{(+)}$ is stable.

We now suppose $A$ is strictly power-associative. Let $e$ be an idempotent of the ring $A$. Then for $x\in A_e(2)$, and $w\in A_e(1)$ we have $x \circ w=w_1+w_0$ where $w_1\in A_e(1)$, $w_0\in A_e(0)$. Then the mapping $w\to w_1$ is an endomorphism of the module $A_e(1)$ determined by the element $x$ of $A_e(2)$. We denote this mapping by $S(x)$. By Albert's result [6, Theorem 1](4) the mapping $x\to 2S(x)$ of the ring $A_e^{(+)}(2)$ onto the special Jordan ring of endomorphisms $S(x)$ is a homomorphism with kernel $B_e$ where $B_e$ is the set of elements $x\in A_e^{(+)}(2)$ such that $x\circ w\in A_e(0)$ for all $w\in A_e(1)$. Certainly $B_e$ is an ideal of $A_e^{(+)}(2)$ and we shall show that $B_e$ is in fact an ideal of $A_e(2)$.

LEMMA 4.1. *If $x\in A_e(2)$ and $w\in A_e(1)$, $xw=a_1+a_0$, $wx=b_1+b_0$, then $a_0=b_0$ and $a_1e=eb_1$.*

**Proof.** The result follows from our earlier remarks on such products.

LEMMA 4.2. *If $x\in A_e(2)$ such that $xw+wx\in A_e(0)$ for all $w\in A_e(1)$, then $xw=wx\in A_e(0)$.*

**Proof.** If $xw+wx\in A_e(0)$, then by Lemma 4.1 we see that $a_1=-b_1$. From our earlier results $a_1-b_1=4d_1$ where $ea_1=a_1/2+d_1$ and $ed_1=d_1e=d_1/2$. Thus $2a_1=a_1-b_1=4d_1$ and $ea_1=a_1e=a_1/2$. But then $d_1=0$ so that $a_1=b_1=0$ and $xw=wx=a_0\in A_e(0)$.

LEMMA 4.3. *$B_e$ is an ideal of $A_e(2)$.*

**Proof.** Let $y$ be an arbitrary element of $A_e(2)$. We set $(xy)w=a_1+a_0$, $w(xy)=a_1'+a_0$, $(yx)w=b_1+b_0$, $w(yx)=b_1'+b_0$. By various substitutions of $x$, $y$, $w$ in (28) we find $(xy)w+(xw)y+w(yx)+y(wx)=2(yx)w+2(wx)y$ and $(yx)w+(yw)x+w(xy)+x(wy)=2(xy)w+2(wy)x$. Since $xw=wx\in A_e(0)$ and $xA_e(1)=A_e(1)x\subseteq A_e(0)$, the first of these equations becomes $(xy)w+w(yx)=2(yx)w$ and the second implies that $(yx)w+w(xy)-2(xy)w\in A_e(0)$. Hence, $a_0=b_0$, $a_1+b_1'=2b_1$, and $b_1+a_1'=2a_1$. Adding we find $a_1'+b_1'=a_1+b_1$ and then using Lemma 4.1 we obtain $ea_1+eb_1=ea_1'+eb_1'=a_1e+b_1e=a_1'e+b_1'e=(a_1+b_1)/2=(a_1'+b_1')/2$. Then $ea_1+eb_1'=2eb_1$ or $ea_1=2eb_1-eb_1'=2eb_1-b_1e$. If $ea_1=a_1/2+d_1$, then $eb_1=b_1/2-d_1$ so that $a_1/2+d_1=b_1-2d_1-b_1/2+d_1$. Thus $(a_1-b_1)/2=-2d_1$ so that $e(a_1-b_1)=(a_1-b_1)e=(a_1-b_1)/2$. Then $e(a_1+b_1)+e(a_1-b_1)=(a_1+b_1)/2+(a_1-b_1)/2=a_1$. Hence $d_1=0$ and $a_1=b_1=a_1'=b_1'$. We note that we have actually shown that $(xy)w=(yx)w=w(xy)=w(yx)$. Now set $z=e$ in (29). Then consideration of the preceding remark yields

(4) Any reference to [6] shall also imply a reference to the corresponding results of [10] for the characteristic 5 case.

$2(xy)w+2((xy)w)e+2(yw)x+(wy)x+((yw+wy)e)x-4x((yw+wy)=0$. Then, using the properties of $x$, we have $2(xy)w+2((xy)w)e\in A_e(0)$. Hence $2a_1+2a_1e=3a_1\in A_e(0)$ and, since the characteristic of $A$ is not three, $a_1=0$. This completes the proof.

We now suppose $A$ to be a simple ring with an identity 1 such that $1=e_1+e_2+e_3$ where the $e_i$ are pairwise orthogonal idempotents.

LEMMA 4.4. *Let $e$ be an idempotent of $A$ such that $e\neq 1$ and suppose $x\in A_e(2)$ with the property that $xw+wx=0$ for all $w\in A_e(1)$. Then $x=0$.*

**Proof.** By Lemmas 4.1 and 4.2 we see that $(xw)_1=(wx)_1=0=(xw)_0=(wx)_0$. Let $C_e$ be the set of all such $x\in A_e(2)$. We claim that $C_e$ is an ideal of $A$ and, since $e\neq 1$ and $e$ is not in $C_e$, it will follow that $C_e=0$. By the above remark $C_eA_e(1)=A_e(1)C_e=0$ and since $A_e(2)A_e(0)=A_e(0)A_e(2)=0$ the proof will be complete if we show that $A_e(2)C_e+C_eA_e(2)\subseteq C_e$. Let $y\in A_e(2)$. Then, as in the proof of Lemma 4.3, we find $(xy)w=w(xy)=w(yx)=(yx)w$. Again, as in the proof Lemma 4.3, a substitution of $x$, $y$, $w$, $e=z$ in (29) yields $2(xy)w+2((xy)w)e=0$. Thus $((xy)w)_0=0$ and $((xy)w)_1+((xy)w)_1e=0$ which is impossible unless $((xy)w)_1=0$. Therefore $xy$, $yx\in C_e$ as was to be shown.

LEMMA 4.5. *Set $g=e_1+e_2$. Then $B_g=B_{e_1}+B_{e_2}$.*

**Proof.** From an earlier remark we may set $A=A_{11}+A_{12}+A_{22}+A_{13}+A_{23}+A_{33}$ where $A_{11}+A_{12}+A_{22}=A_g(2)$, $A_{13}+A_{23}=A_g(1)$, and $A_{33}=A_g(0)$. Suppose $x_{11}+x_{12}+x_{22}\in B_g\subseteq A_g(2)$. Then $x_{12}=4(x_{11}+x_{12}+x_{22})\circ e_1-4((x_{11}+x_{12}+x_{22})\circ e_1)\circ e_1$ and since $B_g$ is an ideal of $A_g(2)$, $x_{12}\in B_g$. Then $e_1(x_{11}+x_{22})=x_{11}\in B_g$ so that $x_{22}\in B_g$ also. Suppose $a_{13}+a_{23}\in A_g(1)$. Then $2x_{12}\circ(a_{13}+a_{23})=2x_{12}\circ a_{13}+2x_{12}\circ a_{23}\in A_{23}+A_{13}$ by our remarks on the multiplication of the $A_{ij}$. But $x_{12}\in B_g$ so that $2x_{12}\circ(a_{13}+a_{23})\in A_g(0)=A_{33}$. Hence $x_{12}\circ(a_{13}+a_{23})=0$. Apply Lemma 4.4 to obtain $x_{12}=0$ and thus $B_g\subseteq A_{11}+A_{22}$. For $a_{12}\in A_{12}$ we find $x_{11}\circ a_{12}\in A_{12}+A_{22}$. But $x_{11}\in B_g$ which is an ideal of $A_g(2)$ so that $x_{11}\circ a_{12}\in A_{22}$. By the definition of $B_g$ we have $x_{11}\circ a_{13}\in A_g(0)=A_{33}$. Thus if $w\in A_{12}+A_{13}=A_{e_1}(1)$, then $x_{11}\circ w\in A_{22}+A_{33}=A_e(0)$. Therefore $x_{11}\in B_{e_1}$ and, in a similar manner, $x_{22}\in B_{e_2}$. Hence $B_g\subseteq B_{e_1}+B_{e_2}$. If $x_{11}\in B_{e_1}$, then $x_{11}\circ(A_{12}+A_{13})\subseteq A_{22}+A_{23}+A_{33}=A_{e_1}(0)$. But then $x_{11}\circ A_g(1)=x_{11}\circ(A_{13}+A_{23})=x_{11}\circ A_{13}\subseteq A_{33}=A_g(0)$. Hence $B_{e_1}\subseteq B_g$ and likewise $B_{e_2}\subseteq B_g$. Thus $B_{e_1}+B_{e_2}=B_g$.

LEMMA 4.6. *$B=B_{e_1}+B_{e_2}+B_{e_3}$ is an ideal of $A$.*

**Proof.** Since $B_g=B_{e_1}+B_{e_2}$ is an ideal of $A_g(2)$, we have $A_g(2)B_{e_1}+B_{e_1}A_g(2)\subseteq B_{e_1}+B_{e_2}$. If $h=e_1+e_3$, then $A_h(2)B_{e_1}+B_{e_1}A_h(2)\subseteq B_{e_1}+B_{e_3}$. Since $A_{23}\subseteq A_{e_1}(0)$, $B_{e_1}A_{23}=A_{23}B_{e_1}=0$. Thus $AB_{e_1}+B_{e_1}A=(A_{11}+A_{12}+A_{22})B_{e_1}+B_{e_1}(A_{11}+A_{12}+A_{22})+(A_{11}+A_{13}+A_{33})B_{e_1}+B_{e_1}(A_{11}+A_{13}+A_{33})+A_{23}B_{e_1}+B_{e_1}A_{23}\subseteq B_{e_1}+B_{e_2}+B_{e_3}$. Interchanging subscripts we find $AB_{e_2}+B_{e_2}A\subseteq B_{e_1}+B_{e_2}+B_{e_3}$ and $AB_{e_3}+B_{e_3}A\subseteq B_{e_1}+B_{e_2}+B_{e_3}$. Therefore $B$ is an ideal of $A$.

Either $B=A$ or $B=0$ since we assumed $A$ to be simple. $B=A$ is impossible since $B\subseteq A_{11}+A_{22}+A_{33}$. Thus we have $B=0$ so that $B_{e_1}=B_{e_2}=0$. Hence

$$A_1^{(+)} = A_{11}^{(+)} + A_{12}^{(+)} + A_{22}^{(+)},$$

$$A_2^{(+)} = A_{11}^{(+)} + A_{13}^{(+)} + A_{33}^{(+)} \text{ and } A_3^{(+)} = A_{22}^{(+)} + A_{23}^{(+)} + A_{33}^{(+)}$$

are Jordan rings. We refer the reader to the proof of Theorem 1 in Albert's paper [6] and note that his combinatorial type proof will suffice to show that $A^{(+)}$ is a Jordan ring. We now state the following:

THEOREM 4.3. *If $A$ is a simple strictly power-associative ring of characteristic prime to six satisfying* (28) *and possessing an identity element which is the sum of three pairwise orthogonal idempotents, then $A$ is Jordan-admissible.*

Now suppose that $e$ is a principal idempotent of the strictly power-associative algebra $A$ which has characteristic prime to six and satisfies (28). Then $e$ is also a principal idempotent of $A$ and by [10, Theorem 5] $A_e(1)+A_e(0)\subseteq$radical of $A^{(+)}$. Hereafter we shall denote the radical of $A^{(+)}$ by Rad $A^{(+)}$. We shall attempt to show that the ideals generated in $A$ and $A^{(+)}$ by $A_e(1)+A_e(0)$ are the same.

LEMMA 4.7. *If $A_e(1)\subseteq$Rad $A^{(+)}$ and $z$ and $w$ are elements of $A_e(1)$, then $zw$ and $wz$ are in the* Rad $A^{(+)}$. *Also* $(zw)_2=2(ez\circ w)_2$.

**Proof.** Substitution of $x=z$, $y=w$, $z=e$ in (28) yields $(ez)w+w(ze)+(ew)z+z(we)=2(we)z+2(ze)w$ or equivalently

$$(31)\qquad zw - (ze)w + w(ze) + wz - (we)z + z(we) = 2(we)z + 2(ze)w.$$

Another substitution of $x=e$, $y=w$ yields $(ze)w+w(ez)+(zw)e+e(wz)=2(wz)e+2(ez)w$ or equivalently

$$(32)\qquad (ze)w + wz - w(ze) + (zw)e + e(wz) = 2(wz)e + 2zw - 2(ze)w.$$

Adding (31) and (32) we obtain

$$(33)\qquad 2wz + (zw)e + e(wz) + z(we) = zw + 2(wz)e + 3(we)z.$$

Set $zw=a_2+a_1+a_0$, $wz=b_2-a_1+b_0$. Equating the $A_e(2)$ components of (33) we obtain $2b_2+a_2+b_2+(z(we))_2=a_2+2b_2+3((we)z)_2$. Thus $b_2=-(z(we))_2+3((we)z)_2$. Let $we=w/2+w'$. Then $b_2=-a_2/2-(zw')_2+3b_2/2+3(w'z)_2$ and so $(a_2-b_2)/2=3(w'z)_2-(zw')_2$. But, if we carry through the same argument with $w$ replaced by $w'$, we find $(w'z)_2=(zw')_2$. Thus $(a_2-b_2)/2=2(w'z)_2$. We note that $2(ew\circ z)_2=((ew)z+z(ew))_2=(wz)_2/2+(zw)_2/2-(w'z)_2-(zw')_2$ so that $2(ew\circ z)_2=(b_2+a_2)/2-2(w'z)_2=(b_2+a_2)/2-(a_2-b_2)/2=b_2$. But $b_2=(wz)_2$. By consideration of the $A_e(0)$ components of (33) we obtain $2b_0+(z(we))_0=a_0+3((we)z)_0$. Hence $2b_0-a_0=3b_0/2-a_0/2+3(w'z)_0-(zw')_0$ or

$(b_0-a_0)/2=3(w'z)_0-(zw')_0$. As before $(w'z)_0=(zw')_0$. Then $2(ew\circ z)_0 =((ew)z+z(ew))_0=(b_0+a_0)/2-2(w'z)_0=(b_0+a_0)/2-(b_0-a_0)/2=a_0=(zw)_0$. Then $2(ew\circ z)=(a_2+b_2)/2+(a_0+b_0)/2-(b_0-a_0)/2-(a_2-b_2)/2=b_2+a_0$ which is in the Rad $A^{(+)}$. Hence $b_2$ and $a_0$ are in Rad $A^{(+)}$ and likewise $a_2$ and $b_0$ are in Rad $A^{(+)}$ so that $zw$ and $wz$ are in Rad $A^{(+)}$.

LEMMA 4.8. *If $x$ is in $A_e(\mu)$ and $w$ is in $A_e(1)$ for $\mu=0, 2$ and if either $x$ is in* Rad $A^{(+)}$ *or $A_e(1)\subseteq$*Rad $A^{(+)}$, *then $xw$ and $wx$ are in* Rad $A^{(+)}$.

**Proof.** Set $we=w/2+w'$. Substitution of $y=w$, $z=e$ in (28) yields

$$x(we)+w(xe)+(ew)x+(ex)w=2(xe)w+2(we)x. \tag{34}$$

Suppose $x\in A_e(2)$. Then from (34) we have $x(we)+wx+(ew)x+xw=2xw+2(we)x$ or $xw-wx=x(we)+(ew)x-2(we)x=x(we)+wx-(we)x-2(we)x =xw/2+xw'+wx-3wx/2-3w'x$ and, hence, $(xw-wx)/2=xw'-3w'x$. But by the same argument with $w$ replaced by $w'$ we find $xw'=w'x$ so that $-4w'\circ x =(xw-wx)/2$. Hence, if $x\in$Rad $A^{(+)}$, then $xw-wx\in$Rad $A^{(+)}$. But $2x\circ w =xw+wx\in$Rad $A^{(+)}$ so that $xw$, $wx\in$Rad $A^{(+)}$. Suppose $x\in A_e(0)$. Then (34) becomes $x(we)+(ew)x=2(we)x$ and $xw/2+xw'+wx/2-w'x=2w'x-wx$ or $(xw-wx)/2=3w'x-xw'=2w'x=4w'\circ x$. Hence, if $x$ is in Rad $A^{(+)}$, then so are $xw$ and $wx$. The result is clear if we replace the condition that $x\in$ Rad $A^{(+)}$ by the condition that $A_e(1)\subseteq$Rad $A^{(+)}$.

LEMMA 4.9. *The ideal generated in $A$ by $A_e(1)+A_e(0)$ is contained in* Rad $A^{(+)}$ *if $A_e(1)+A_e(0)\subseteq$*Rad $A^{(+)}$.

**Proof.** We shall show that the ideal in $A$ generated by $A_e(1)+A_e(0)$ is contained in $N+A_e(1)+A_e(0)$ where $N$ is defined as the set of all finite linear combinations of elements of the form $(x_1x_0)_2$ and $(x_1z_1)_2$. By our multiplication we see that it is sufficient to show that $NA_e(2)+A_e(2)N\subseteq N$. Set $L=N+A_e(1)+A_e(0)$. Let $y$, $z\in A_e(1)$ and $x\in A_e(2)$. Then (28) reads $x(yz)+y(xz)+(zy)x+(zx)y=2(xz)y+2(yz)x$. Thus $x(yz)+(zy)x-2(yz)x\in L$. Interchanging $y$ and $z$ we have $x(zy)+z(xy)+(yz)x+(yx)z=2(xy)z+2(zy)x$ so that $x(zy)+(zy)x-2(zy)x\in L$. Adding these two expressions we find $x(yz+zy)-(yz+zy)x\in L$. Now in (29) we substitute $x, y, z, w=e$ obtaining $2x(zy+yz)+2(zy+yz)x+y(xz+zx)+(xz+zx)y+z(xy+yx)+(xy+yx)z=((xy+yx)z)e+((xy+yx)e)z+((xz+zx)e)y+((xz+zx)y)e+2(xy)z+2(xz)y+(yz)x+(yx)z+(zy)x+(zx)y+((zy+yz)e)x+((zy+yz)x)e$. By the restrictions on $x$, $y$, $z$ and the assumption that $A_e(1)+A_e(0)\subseteq L$ we find that $2x(zy+yz)-(zy+yz)x\in L$. From above $x(yz+zy)-(yz+zy)x\in L$ and hence $x(y\circ z)$, $(y\circ z)x\in L$. Thus $2x(ey\circ z)_2\in L$ and by Lemma 4.7, $x(yz)_2\in L$.

Now let $x\in A_e(2)$, $y\in A_e(1)$, and $z\in A_e(0)$. Set $yz=a_2+a_1$, $zy=a_2+b_1$. Then substitution in (28) yields $x(yz)+(zy)x=2(yz)x$ or $xa_2+a_2x=2a_2x$. Thus $a_2x=xa_2$. Substitution of $x, y, z, w=e$ in (29) yields $2x(yz+zy)+2(zy+zy)x =((xy+yx)z)e+((xy+yx)e)z+2(xy)z+(yz)x+(yz)x+(yx)z+((zy+yz)x)e$

$+ ((zy + yz)e)x$. Hence $2x(yz + zy) + 2(yz + zy)x - (yz)x - ((zy + yz)x)e - ((zy+yz)e)x \in L$ so that $2x(yz+zy) - (yz)x \in L$. Thus $2x(a_2+a_2) - a_2x = 3xa_2 \in L$. Since the characteristic of $A$ is prime to six, $xa_2 = x(yz)_2 \in L$. Thus the ideal generated by $A_e(1)+A_e(0)$ is contained in $N+A_e(1)+A_e(0)$. Therefore this ideal must be equal to $N+A_e(1)+A_e(0)$.

If $e$ is a principal idempotent of $A$, then $e$ is also a principal idempotent of $A^{(+)}$ and, as we have stated earlier, $A_e(1)+A_e(0) \subseteq \text{Rad } A^{(+)}$. By the preceding Lemma $N+A_e(1)+A_e(0)$ is an ideal of $A$ which is contained in Rad $A^{(+)}$. Hence $N+A_e(1)+A_e(0)$ is a nil ideal of $A$. If we now suppose $A$ to be semi-simple, then $N+A_e(1)+A_e(0) = 0$ so that $A = A_e(2)$. Therefore $e$ is an identity for $A$ and we state this as

THEOREM 4.4. *Every semi-simple strictly power-associative algebra of characteristic prime to six satisfying* (28) *has an identity element.*

Suppose $D$ is an ideal of a semi-simple strictly power-associative algebra $A$ of characteristic prime to six satisfying (28). If $D \neq 0$, then since $D$ is not nil $D$ must possess an idempotent $e$ and we may suppose $e$ to be principal. Then $D = D_e(2)+D_e(1)+D_e(0)$. Since $e \in D$, we must have $A_e(2)$ and $A_e(1)$ contained in $D$. Hence $D_e(2) = A_e(2)$ and $D_e(1) = A_e(1)$ so that we may write $D = A_e(2)+A_e(1)+D_e(0)$. Let $M$ be the radical of $D$. Then since $e$ is principal, $D_e(1)+D_e(0) \subseteq M$. In order that $M$ be an ideal of $A$ we see that it is sufficient to show that $A_e(0)M+MA_e(0) \subseteq M$. Let $x \in A_e(0)$ and $m = m_2+m_1+m_0 \in M$. Then $xm = xm_1+xm_0$ where $xm_1 \in A_e(2)+A_e(1)$ and $xm_0 \in D_e(0)$ since $D$ is an ideal of $A$. Hence, it is sufficient to show that $(xm_1)_2 \in M$. Since $(xm_1)_2 = (m_1x)_2$ we need only show $(x \circ m_1)_2 \in M$. We note that (29) holds in $A^{(+)}$ which is commutative. Setting $x = y$ and $z = w = m_1$ in (29) for $A^{(+)}$ we find $8x^2 \circ m_1^2 + 16(x \circ m_1)^2 = 4(x^2 \circ m_1) \circ m_1 + 4(m_1^2 \circ x) \circ x + 8((x \circ m_1) \circ x) \circ m_1 + 8((x \circ m_1) \circ m_1) \circ x$. Thus we have $4(x \circ m_1)^2 - 2((x \circ m_1) \circ m_1) \circ x = (x^2 \circ m_1) \circ m_1 + (m_1^2 \circ x) \circ x + 2((x \circ m_1) \circ x) \circ m_1 + 2x^2 \circ m_1^2$. Remembering that $m_1 \in M \cap D_e(1) = D_e(1)$ and $D_e(1)+D_e(0) \subseteq M$ we obtain $(x^2 \circ m_1) \circ m_1 \in M$, $(m_1^2 \circ x) \circ x \in A_e(0) \cap D = D_e(0) \subseteq M$, $2((x \circ m_1) \circ x) \circ m_1 \in D \circ M \subseteq M$, and $2x^2 \circ m_1^2 \in A_e(0) \cap D = D_e(0) \subseteq M$. Thus the right-hand member of the above equation is an element of $M$ so that $2(x \circ m_1)^2 - ((x \circ m_1) \circ m_1) \circ x \in M$. Then the $D_e(2)$ component of this expression is in $M$. Set $x \circ m_1 = b_2+b_1$. Then $2(b_2^2+2b_2 \circ b_1+b_1^2) - (b_2 \circ m_1) \circ x - (b_1 \circ m_1) \circ x \in M$ and the $D_e(2)$ component is $2b_2^2+2(b_1^2)_2 - ((b_2 \circ m_1) \circ x)_2$. Using [6, Identity 8] we obtain $((b_2 \circ m_1) \circ x)_2 = ((b_2 \circ m_1)_1 \circ x)_2 = (m_1 \circ x)_2/2 \circ b_2$. But $(m_1 \circ x)/2 \circ b_2 = b_2^2/2$. Therefore $3b_2^2/2 - 2(b_1^2)_2 \in M$. But $(b_1^2)_2 \in M$ since $b_1 \in D_e(1) \subseteq M$. Thus since the characteristic of $A$ is not three, $b_2^2 \in M$. Every element of $M$ is nilpotent so that $b_2$ is nilpotent. Now consider the ideal of $D$ generated by $M$ and all elements of the form $b_2 = (xm_1)_2$. If we can show that $y(xm_1)_2$ is of the form $(xm_1)_2$ for all $y \in A_e(2) = D_e(2)$, then this ideal will be a nil ideal of $D$ containing the maximal nil ideal $M$ leaving $b_2 = (xm_1)_2 \in M$ as the only remaining possibility. Substitution of $x$, $y$, $m=z$, and $w=e$ in (29) yields $2y(xm_1+m_1x) + 2(xm + m_1x)y = 2(ym_1)x + ((ym_1 + m_1y)x)e + ((ym_1 + m_1y)e)x + (m_1y)x + (m_1x)y + ((m_1x+xm_1)y)e + ((m_1x+xm_1)e)y$. Considering the $A_e(2)$ components and using $(y(xm_1))_2 = (y(m_1x))_2 = ((m_1x)y)_2 = ((xm_1)y)_2$ we find $8(y(xm_1))_2 = 2((ym_1)x)_2 + ((ym_1 + m_1y)x)_2 + (((ym_1 + m_1y)e)x)_2 + ((m_1y)x)_2 + (y(xm_1))_2 + 2(y(xm_1))_2 + 2(y(m_1x))_2$. Simplifying we obtain $3y(xm_1)_2 = ((3ym_1+2m_1y+(ym_1+m_1y)e)_1x)_2$ and we observe that

$$(3ym_1 + 2m_1y + (ym_1 + m_1y)e)_1 \in M \cap A_e(1).$$

Thus, since the characteristic of $A$ is not three, $y(xm_1)_2$ is of the desired form and, hence, is nilpotent. Therefore $M$ is a nil ideal of $A$ which we had assumed to be semi-simple; therefore $M = A_e(1) = D_e(0) = 0$. Thus $A = A_e(2) \oplus A_e(0)$ where $A_e(2)$ and $A_e(0)$ are semi-simple algebras with identity elements $e$ and $1-e$ respectively. Proceeding in the usual manner we may now state

THEOREM 4.5. *Every semi-simple strictly power-associative algebra A over a field F of characteristic prime to six and satisfying* (28) *has an identity and is the direct sum of simple algebras.*

In the following we shall suppose $A$ to be a strictly power-associative algebra over a field $F$ of characteristic prime to six such that $A$ is simple and satisfies (28). Moreover, we assume that $A$ has an identity 1 which can be written as $1=e_1+e_2$ where the $e_i$ are pairwise orthogonal idempotents of $A$ and that $A$ is Jordan-admissible. $A$ is simple over its center and simple over the algebraic closure $K$ of its center. Thus $A_K$ is a simple algebra which is Jordan-admissible and we may suppose $F=K$. Then we can set $1=u+v$ where $u$ is a primitive idempotent of $A$. Since $A_u^{(+)}(2)$ is a Jordan algebra, we may use [8, Theorem A] to write $A_u(2) = uK+N$ where $N$ is the ideal of nilpotent elements of $A_u^{(+)}(2)$. Suppose $N$ is not an ideal of $A_u(2)$. Then there are elements $x, y \in N$ such that $xy = u+n$ and $yx = -u+n'$ where $n, n' \in N$. Suppose $w \in A_u(1)$. Then substitution of $x=w$, $z=x$ in (28) yields $(xy)w + (xw)y + y(wx) + w(yx) = 2(yx)w + 2(wx)y$. Then $uw + nw + (xw)y + y(wx) - wu + wn' = -2uw + 2n'w + 2(wx)y$. Rearranging terms we find $3uw - wu = 2n'w - nw - wn' + 2(wx)y - y(wx) - (xw)y$. Since $A^{(+)}$ is a Jordan algebra, it is stable [2] and by our earlier remark, $A$ is stable. Then applying Lemma 4.8 to each term of the right-hand member of the above relation we find that the right-hand member is in Rad $A^{(+)}$ and, consequently, the left-hand member is also in Rad $A^{(+)}$. Now we set $uw = w/2 + w'$ where $w'u = uw' = w'/2$. Since $w$ was an arbitrary element of $A_u(1)$, $3uw' - w'u = w' \in$ Rad $A^{(+)}$. But then $3uw - wu = 3w/2 + 3w' - w/2 + w'$ so that $w \in$ Rad $A^{(+)}$. Thus $A_u(1) \subseteq$ Rad $A^{(+)}$ and an application of Lemma 4.7 yields $A_u(1) + A_u(1)A_u(1) \subseteq$ Rad $A^{(+)}$. We claim $A_u(1) + A_u(1)A_u(1)$ is, in fact, an ideal of $A$. Let $x \in A_u(2)$ and $y, z \in A_u(1)$. Substitution of $y=e$ in (29) yields $2x(wz+zw) + 2(wz+zw)x + w(zx+xz) + (zx+xz)w + z(xw+wx) + (xw+wx)z = 2(xw)z$

$+ 2(xz)w + ((xz + zx)e)w + ((xz + zx)w)e + ((xw + wx)e)z + (xw + wx)z)e + ((zw + wz)e)x + ((zw + wz)x)e + (wz)x + (wz)x + (wx)z + (zw)x + (zx)w$. Using the stability of $A$ we find $2x(zw+wz)-(zw+wz)x \in A_u(1)A_u(1)$. The two different substitutions of $y=w$ and $y=z$, $z=w$ in (28) yield the following relations: $x(wz)+(zw)x-2(wz)x \in A_u(1)A_u(1)$ and $x(zw)+(wz)x-2(zw)x \in A_u(1)A_u(1)$. Adding these two we have $x(wz+zw)-(wz+zw)x \in A_u(1)A_u(1)$. Combining this with the above remark that $2x(zw + wz) - (wz + zw)x \in A_u(1)A_u(1)$, we find $x(wz+zw) \in A_u(1)A_u(1)$. Replacing $z$ by $uz$ and applying Lemma 4.7, we see that $x(zw)_2 = x(uz \circ w) \in A_u(1)A_u(1)$. This result along with the stability of $A$ makes $A_u(1)+A_u(1)A_u(1)$ closed under multiplication by $A_u(2)$. We note that $A_v(2)=A_u(0)$ and $A_v(0)=A_u(2)$. Interchanging the roles of $u$ and $v$, we find that $A_u(1)+A_u(1)A_u(1)=A_v(1)+A_v(1)A_v(1)$ is closed under multiplication by $A_u(0)$. Noting the stability of $A$ it is clear that $A_u(1)+A_u(1)A_u(1)$ is closed under multiplication by $A_u(1)$. Hence, $A_u(1)+A_u(1)A_u(1)$ is an ideal of $A$ which is contained in Rad $A^{(+)}$ and, since the simple algebra $A$ contains an idempotent, we must have $A_u(1)=0$. But then $A=A_u(2)\oplus A_u(0)$ which is impossible by the simplicity of $A$. Thus $N$ is an ideal of $A_u(2)$.

Now suppose $1=e_1+ \cdots +e_t$ where the $e_i$ are primitive pairwise orthogonal idempotents of $A$. Then for an arbitrary element $x$ of $A$ we may write $x=\sum k_ie_i+\sum x_{ij}+\sum x_i'$ where $k_i \in K$, $x_{ij}\in A_{ij}$ for $i\neq j$, and $x_i' \in N_i$ where $N_i$ is the ideal of nilpotent elements of $A_{e_i}(2)$. Setting $\delta(x)=\sum k_i$ defines a linear function on $A$ to $K$. We show that the following conditions are satisfied by $\delta$:

(1) $\delta(xy)=\delta(yx)$;

(2) $\delta(x(yz))=\delta((xy)z)$.

We shall first show that (1) is satisfied by $\delta$. Let $x=\sum k_ie_i+\sum x_{ij}+\sum x_i'$ and $y=\sum k_i' e_i+\sum y_{ij}+\sum y_i'$. Since $N_i$ is an ideal of $A_{e_i}(2)$, $x_iy_i\in N_i$. By the stability of $A$, for $i\neq j$, $k$ arbitrary, we have $x_{ij}y_k$, $y_kx_{ij}\in A_{ij}$. Also $x_{ij}y_{jn} \in A_{in}$ for $i$, $j$, $n$ distinct. Hence, $\delta(xy)-\delta(yx)=\delta[\sum x_{ij}y_{ij}]-\delta[\sum y_{ij}x_{ij}]$, so that by the linear property of $\delta$ it is sufficient to show that $\delta(x_{ij}y_{ij})=\delta(y_{ij}x_{ij})$. Set $x_{ij}y_{ij}=xy=a_2+a_1+a_0$ and $y_{ij}x_{ij}=yx=b_2-a_1+b_0$ where $a_2$, $b_2\in A_{ii}$ and $a_0$, $b_0 \in A_{jj}$. Using (28) we find $(xy)e_i+e_i(yx)+(xe_i)y+y(e_ix)=2(yx)e_i+2(e_ix)y$. Set $xe_i=x/2+g$ where $e_ig=ge_i=g/2$. Then we have from above $(xy)e_i+e_i(yx)+xy/2+gy+yx/2-yg=2(yx)e_i+xy-2gy$. Considering only the $A_{ii}$ and $A_{jj}$ components we obtain $a_2+b_2+a_2/2+a_0/2+gy+b_2/2+b_0/2-yg=2b_2+a_2+a_0-2gy$ or $(a_2-b_2)/2+(b_0-a_0)/2=(yg)_2+(yg)_0-3(gy)_2-3(gy)_0=-2(yg)_2-2(yg)_0$. Thus $(b_2-a_2)/2=2(yg)_2$ and $(a_0-b_0)/2=2(yg)_0$. Now $2(e_ig)\circ y=(e_ig)y+y(e_ig)=gy/2+yg/2=(gy)_2+(gy)_0$. But by [6, Lemma 10], $2(e_ig)\circ y=k'(e_i+e_j)+n_i'+n_j'$ so that $\delta((gy)_2)=k'=\delta((gy)_0)$. Then, from above, we see that $\delta(b_2-a_2)=\delta(a_0-b_0)$ or $\delta(a_2)+\delta(a_0)=\delta(b_2)+\delta(b_0)$. Again by [6, Lemma 10] we obtain $xy+yx=k(e_i+e_j)+n_i+n_j$ so that $\delta(a_2+b_2)=\delta(a_0+b_0)$. Hence $\delta(a_2)=\delta(b_0)$ and $\delta(a_0)=\delta(b_2)$. Finally, $\delta(xy)=\delta(a_2)+\delta(a_0)=\delta(b_2)+\delta(b_0)=\delta(yx)$.

We next consider condition (2). We must show that $\delta(x(yz))=\delta((xy)z)$. We restate (28) as $x(yz)+y(xz)+(zy)x+(zx)y=2(xz)y+2(yz)x$. Using $\delta(xy)=\delta(yx)$ we obtain $\delta(x(yz))+\delta(y(xz))-\delta((zy)x-\delta((zx)y)=0$. Hence $\delta(y(zx))-\delta((yz)x)=\delta((xz)y)-\delta(x(zy))$ or $\delta[y(zx)-(yz)x]=\delta[(xz)y-x(zy)]$. Then

$$\begin{aligned}
&4\delta[x\circ(y\circ z)-(x\circ y)\circ z]\\
&\quad=\delta[x(yz)+x(zy)+(yz)x+(zy)x-(xy)z-z(yx)-z(xy)-(yx)z]\\
&\quad=\delta[x(yz)-(xy)z]+\delta[x(zy)-(xz)y]+\delta[(yz)x-y(zx)]+\delta[(zy)x-z(yx)]\\
&\quad=4\delta[x(yz)-(xy)z]
\end{aligned}$$

(by use of $\delta(xy)=\delta(yx)$ and $\delta[y(zx)-(yz)x]=\delta[(xz)y-x(zy)]$). Thus it is sufficient to show that (2) holds in $A^{(+)}$. Let the coefficient of $e_i$ in the decomposition of $x$, $y$, $z$ be respectively $\alpha_i$, $\beta_i$, $\gamma_i$. Then

$$\begin{aligned}
2\delta[z\circ(x\circ y)]=2\sum\alpha_i\beta_i\gamma_i&+2\delta[\sum(\gamma_ie_i+\gamma_je_j)\circ(x_{ij}\circ y_{ij})]\\
&+\delta[\sum(\alpha_i+\alpha_j)z_{ij}\circ y_{ij}]+\delta[\sum(\beta_i+\beta_j)z_{ij}\circ x_{ij}]\\
&+2\delta[\sum z_{ik}\circ(x_{ij}\circ y_{ik})].
\end{aligned}$$

From the proof of (1) we see that $\delta[e_i\circ(x_{ij}\circ y_{ij})]=\delta(x_{ij}\circ y_{ij})/2=\delta[e_j\circ(x_{ij}\circ y_{ij})]$. Hence, if we consider the above expression with $x$ and $z$ interchanged, we observe that it will suffice to show that $\delta[z_{ik}\circ(x_{ij}\circ y_{jk})]=\delta[x_{ij}\circ(z_{ik}\circ y_{jk})]$ for $i$, $j$, $k$ distinct.

We now state the Jordan identity for $A^{(+)}$ in its linearized form.

$$(35)\qquad \sum(x\circ y)\circ(w\circ z)=\sum((x\circ y)\circ w)\circ z \qquad \text{(symmetric in } x, y, z).$$

Set $x=x_{ij}$, $y=y_{jk}$, $w=z_{ik}$, and $z=e_i$. Then we find $(x_{ij}\circ y_{jk})\circ z_{ik}+(z_{ik}\circ y_{jk})\circ x_{ij}=2((x_{ij}\circ y_{jk})\circ z_{ik})\circ e_i+(x_{ij}\circ z_{ik})\circ y_{jk}$. Then interchanging $x$ and $z$, and $i$ and $k$ we have $(z_{ik}\circ y_{jk})\circ x_{ij}+(x_{ij}\circ y_{jk})\circ z_{ik}=2((z_{ik}\circ y_{jk})\circ x_{ij})\circ e_i+(z_{ik}\circ x_{ij})\circ y_{jk}$. Subtracting we have $((x_{ij}\circ y_{jk})\circ z_{ik})\circ e_i=((z_{ik}\circ y_{jk})\circ x_{ij})\circ e_i$. Then, using the above remarks we obtain $\delta[((x_{ij}\circ y_{jk})\circ z_{ik})\circ e_i]=\delta[((x_{ij}\circ(y_{jk}\circ z_{ik})\circ e_i]=\delta[x_{ij}\circ(y_{jk}\circ z_{ik})]/2=\delta[(x_{ij}\circ y_{jk})\circ z_{ik}]/2$. Therefore (2) holds. By (1) and (2) the set $N_\delta$ of all $x\in A$ such that $\delta(xy)=0$ for all $y\in A$ is an ideal of $A$. Surely Rad $A^{(+)}\subseteq N_\delta$ and, since $A$ is simple and $\delta(e_1)\neq 0$, $N_\delta=\text{Rad }A^{(+)}=0$. Thus $A^{(+)}$ is a simple Jordan algebra [3, Chapter V, Theorem 8](5).

THEOREM 4.6. *If $A$ is a simple strictly power-associative algebra over field $K$ of characteristic $\neq 2$, 3 which satisfies* (28) *and such that $A^{(+)}$ is a Jordan algebra of degree $t>1$, then $A^{(+)}$ is a simple Jordan algebra.*

We now reproduce Albert's argument [4] to show that under the hypotheses of the above theorem $A$ is flexible. Set $x=z$ in (2). Then $w=(xy+yx)x$

(5) Albert's result is for flexible algebras but holds equally well for this case since the $A_{ii}$ are orthogonal subalgebras of $A$.

$-x(xy+yx)+x^2y-yx^2=0$ and so $wz=zw=0$ for all $z\in A$. Interchanging $y$ and $z$ and adding we obtain $((xy)x)z+((yx)x)z-(x(xy))z-(x(yx))z+(x^2y)z-(yx^2)z+((xz)x)y+((zx)x)y-(x(xz))y-(x(zx))y+(x^2z)y-(zx^2)y=0$. Then applying $\delta$ and using properties (1) and (2) repeatedly we find $2\delta[(xy)(xz)]=2\delta[(zx)(yx)]$. Applying (2) to the left-hand member and (2) and (1) to the right-hand member we find $\delta[((xy)x)z]=\delta[(x(yx))z]$ so that $\delta[((xy)x-x(yx))z]=0$ for all $x, y, z\in A$. Hence $(xy)x-x(yx)\in N_\delta$ and since $A$ is simple we are finished.

THEOREM 4.7. *A simple strictly power-associative algebra A over a field K of characteristic* $\neq 2, 3$ *which satisfies* (28) *is*

(a) *a commutative Jordan algebra*;

(b) *a quasi-associative algebra*;

(c) *an algebra of degree* 2; *or*

(d) *an algebra of degree* 1.

**Proof.** The proof is immediate from the preceding remark, Theorems 4.3 and 4.6 and [13, Theorem 4.2].

5. **The identities** $R_x^2+L_x^2=L_xR_x$ **and** $(R_x+L_x)(R_y+L_y)=(R_y+L_y)(R_x+L_x)$. We shall present in this section some examples of algebras satisfying (9) and (11). First consider the algebra $A$ over a field $F$ of characteristic zero with a basis $e_1$, $a$, and $e_2$ where the $e_i$ are orthogonal idempotents such that $e_1+e_2=1$, $e_1a=ae_2=1+a/2$, $e_2a=ae_1=-1+a/2$, and all the remaining products are zero. In order to show that $A$ is power-associative we must show $xx^2=x^2x$ and $(x^2x)x=(x^2)^2$ for all $x\in A$. Set $x=\alpha_1e_1+\alpha a+\alpha_2e_2$. Then $x^2=\alpha_1^2e_1+\alpha(\alpha_1+\alpha_2)a+\alpha_2^2e_2$ and $x^2x=\alpha_1^3e_1+\alpha(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)a+\alpha_2^3e_2=xx^2$. We see that $(x^2)^2=\alpha_1^4e_1+\alpha(\alpha_1+\alpha_2)(\alpha_1^2+\alpha_2^2)a+\alpha_2^4e_2$ and $(x^2x)x=(\alpha_1^3e_1+\alpha(\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)a+\alpha_2^3e_2)(\alpha_1e_1+\alpha a+\alpha_2e_2)=\alpha_1^4e_1+\alpha(\alpha_1+\alpha_2)(\alpha_1^2+\alpha_2^2)a+\alpha_2^4e_2=(x^2)^2$. This along with [1, Lemma 4] implies that $A$ is power-associative. We set $x=\alpha_1e_1+\alpha a+\alpha_2e_2$ and $y=\beta_1e_1+\beta a+\beta_2e_2$. Then $xy=(\alpha_1\beta_1-\alpha\beta_1+\alpha_1\beta-\alpha_2\beta+\alpha\beta_2)e_1+(\alpha\beta_1+\alpha_1\beta+\alpha_2\beta+\alpha\beta_2)a/2+(\alpha_2\beta_2-\alpha\beta_1+\alpha_1\beta-\alpha_2\beta+\alpha\beta_2)e_2$ and $yx=(\alpha_1\beta_1+\alpha\beta_1-\alpha_1\beta+\alpha_2\beta-\alpha\beta_2)e_1+(\alpha\beta_1+\alpha_1\beta+\alpha_2\beta+\alpha\beta_2)a/2+(\alpha_2\beta_2+\alpha\beta_1-\alpha_1\beta+\alpha_2\beta-\alpha\beta_2)e_2$. Hence $xy-yx=2(\alpha_1\beta-\alpha\beta_1+\alpha\beta_2-\alpha_2\beta)e_1+2(\alpha_1\beta-\alpha\beta_1+\alpha\beta_2-\alpha_2\beta)e_2=2(\alpha_1\beta-\alpha\beta_1+\alpha\beta_2-\alpha_2\beta)1$. Thus $xy-yx$ commutes with all $z\in A$ so that $z(xy-yx)=(xy-yx)z$ and (11) holds.

Suppose $L$ is an ideal of $A$ and $x=\alpha_1e_1+\alpha a+\alpha_2e_2\neq 0\in L$. Then $e_1x+xe_1=2\alpha_1e_1+\alpha a\in L$. Thus $2\alpha_1e_1+\alpha a-x=\alpha_1e_1-\alpha_2e_2\in L$. If either $\alpha_1$ or $\alpha_2$ is not zero then by the orthogonality of the $e_i$ either $e_1$ or $e_2\in L$. But then $e_ia+ae_i=a\in L$ implying that $e_1a-a/2=1\in L$. Thus $L=A$. Suppose $\alpha_1=\alpha_2=0$. Then $\alpha\neq 0$ so that $a\in L$ and, as above, $1\in L$. Therefore $A$ is a simple power-associative algebra satisfying (11) which is not flexible since $e_1(ae_1)=-e_1/2+a/4+e_2/2$ and $(e_1a)e_1=e_1/2+a/4-e_2/2$.

We may construct new examples by setting $A$ equal to the algebra over a field $F$ of characteristic zero with a basis $e_1, \cdots, e_n, a_{ij}, i<j=1, \cdots, n,$ where $1=e_1+\cdots+e_n$, the $e_i$ are pairwise orthogonal idempotents, $e_ia_{ij}=a_{ij}e_j=1+a_{ij}/2$, $e_ja_{ij}=a_{ij}e_i=-1+a_{ij}/2$, and all other products are zero. We note that the nonflexible examples given earlier which satisfy (5) also satisfy (11). This is not too surprising when we observe that any algebra $A$ which satisfies (5) and either (9) or (11) must satisfy all three.

Identities (9) and (11) are not strong enough conditions in themselves to enable us to obtain any significant results concerning algebras which satisfy either of them. It is not evident to us at this time what other conditions we might impose on these algebras.

## Bibliography


1. A. A. Albert, *On the power-associativity of rings*, Brasil. Math. **2** (1948), 1–13.
2. ———, *A structure theory for Jordan algebras*, Ann. of Math. **48** (1947), 546–567.
3. ———, *Power-associative rings*, Trans. Amer. Math. Soc. **64** (1948), 552–597.
4. ———, *A theory of trace-admissible algebras*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 317–322.
5. ———, *Almost alternative algebras*, Portugal. Math. **8** (1949), 23–36.
6. ———, *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. **69** (1950), 503–527.
7. ———, *The structure of right alternative algebras*, Ann. of Math. **59** (1954), 407–417.
8. N. Jacobson, *A theorem on the structure of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 140–147.
9. Erwin Kleinfeld, *Rings of* $(\gamma, \delta)$ *type*, Portugal. Math. **18** (1958), 107–110.
10. L. A. Kokoris, *New results on power-associative algebras*, Trans. Amer. Math. Soc. **77** (1954), 363–373.
11. ———, *On a class of almost alternative algebras*, Canad. J. Math. **8** (1956), 250–255.
12. ———, *On rings of* $(\gamma, \delta)$ *type*, Proc. Amer. Math. Soc. **9** (1958), 897–904.
13. Robert H. Oehmke, *On flexible algebras*, Ann. of Math. **68** (1958), 221–230.
14. R. D. Schafer, *Noncommutative Jordan algebras of characteristic* 0, Proc. Amer. Math. Soc. **6** (1955), 472–475.



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